2. Elastic Behaviour: More Than Just Rubber Bands

LEARNING GOALS

Learning Objectives

1. Define Hooke's law, engineering stress, and engineering strain
2. Explain how elastic stress and strain are related through Hooke's law
3. Explain how stress, and strain are sample size independent, while force and displacement are not
4. Describe the conventional uniaxial tensile test
5. Identify and explain the following on a typical "dog-bone" tensile specimen:
    1. reduced section
    2. gauge length
    3. cross sectional area
    4. grip region
6. Describe elastic deformation in terms of macroscopic behaviour as well as in terms of atomic positions
7. Sketch a generalized interatomic force-vs.-separation curve for two atoms in a solid
8. Using the interatomic force-separation curve, describe how the Young's Modulus is a structure independant property

By the time you've lived a few years on the earth, say 10 years or so, you'll have several years of experience interacting with matter and have a fairly good idea of how materials will behave. For example, you know that if you drop a ceramic coffee mug onto a concrete floor the coffee mug will break. If you were to drop the same mug onto a vinyl floor there is a chance that you'll only need to clean up the coffee.

Example: Kite-Boarding Control Lines and Kevlar

Imagine that you want to build a huge kite that you can control and steer and use it to pull you around on a board on the water, or even a buggy on the land. You want this to be exciting, so you want the kite to pull you hard (that is, with a lot of force) so you want the kite to be lightweight. Also, you'll be controlling it from the water so you need to be able to pull on the control line and have the kite respond quickly, meaning that you need a stiff (technically, high *elastic modulus) line. Imagine if the line was made from a bungee cord. Imagine pulling a car with a bungee cord. You could walk a meter and still not have enough force in the cord to move the car. In plain (but imprecise) language, you need a line that doesn't stretch out too much. Wow, you've just designed Spectra® fibre, which is used in this application. The material in Spectra® fibre is manufactured in such a way that the individual bonds between atoms are lined up along the axis of the fibre, giving a high strength, "stiff," lightweight material. The same concept is applied with Kevlar® that is used in bullet resistant armour, and you guessed it, the same concept is applied to the ropes used in some competitive sailing dinghies, and some fishing line.*

The Need For Some Vocabulary

Alright. You've no doubt noticed some of the awkward wording so far: bendy, stiff, stretch. Learning any new topic requires the student to learn a few new words and learning some new words also requires the learning of some new concepts. In this section we'll explore the precise meaning and use of these words:

- (Engineering) Stress
- (Engineering) Strain
- Young's Modulus
- Strength
- Hooke's law

Precise words allow efficient communication. However, until I have defined each of these terms, I will use common terminology. Since these common terms are not entirely correct, I'll put them in quotes, as in the following sentence. A peanut is a "nut." Well, actually, it's a legume.

Hooke's Law: A Fancy Name for a Straight Line

Video

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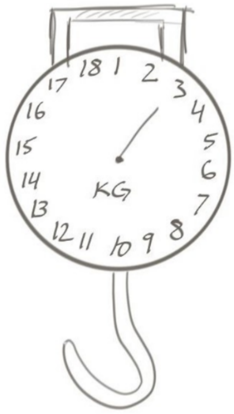

Figure 1: A spring scale, such as those used to weigh fish, luggage, produce

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Perhaps you've used a spring scale (Figure 1) to weigh your luggage before heading to the airport or maybe to weigh a fish or to weigh your tomatoes at the produce section of your grocery store. Inside the spring scale is, you guessed it, a spring. The spring is a spiral wound strip of metal, likely high strength steel. As a load is applied to the hook, the spring is caused to coil up a little more tightly, but because the spring is actually quite long, any given part of the spring is only bending a very small amount (experiencing a small strain).

Because the "bending" is small, the spring doesn't bend permanently, which is good since you would like to use it again. Also, because the "bending" is small it takes advantage of a common property of many materials - especially metals - the relationship between stress and strain (for relatively low values of both) is linear. That is, if you were to "stretch" a material out by applying a range of different loads to it and you recorded on a graph the length of the material for each load you would plot a straight line, as shown in Figure 2.

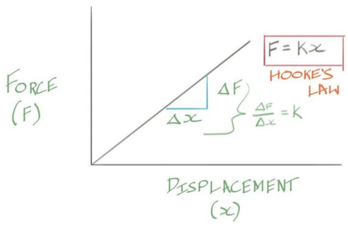

Figure 2: A general linear force versus displacement plot.

The equation of a straight line is often written like this: $y = mx + b$ (1)

where m is the slope, x is abscissa value, and b is the intercept with the y axis. In our case, if there is no force, there is no displacement, so b is zero and the equation, in terms of force F and displacement x, becomes

$$F = kx \tag{2}$$

where k is the slope. The letter k is commonly used for the slope, but you may come across other letters. It is frequently called the spring constant. The equation for a straight line relationship between force and displacement is called Hooke's Law. The problem with using force and displacement to study materials is that the behaviour depends on how big the sample is that we are testing.

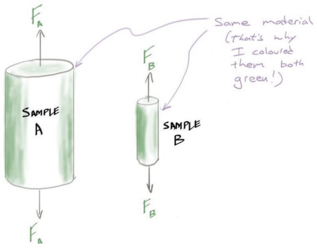

Figure 3: Two samples of the same material both loaded along their long axes.

Consider two cylindrical samples of the same material, as in Figure 3. Both samples are made from exactly the same material, yet you need more force to cause the same elongation in sample A versus sample B. You may be tempted to say that sample A is stronger, but that would be inaccurate and misleading, since they are both made from the same material. It would be like me saying that the apples that I bought were better than yours simply because I bought more of them.

The problem becomes most obvious when we plot the force and displacement data, as we did in Figure 2, except this time for both sample A and sample B. We will find that sample A appears "stronger" (in fact, the material is not stronger, although the sample is capable of supporting a higher load) and sample A also has a higher spring constant. Again, the apples that I bought are more delicious and juicy simply because I bought more of them than you did. That's just not right. To fix this, we need stress and strain.

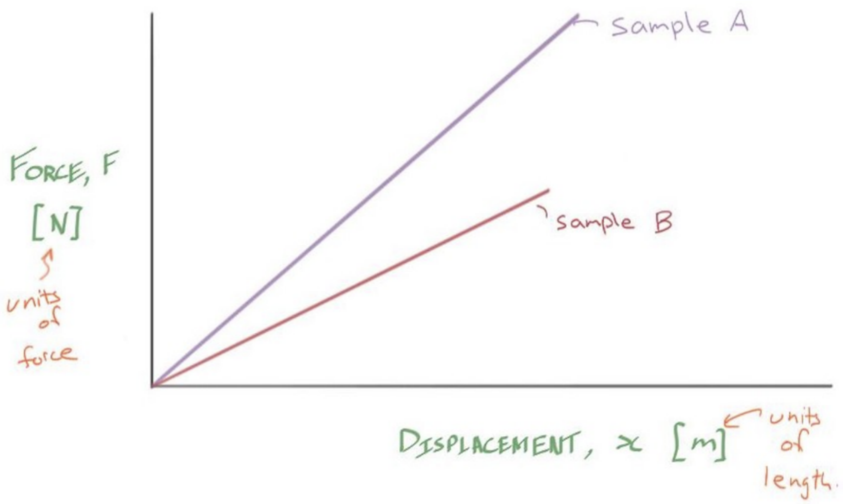

Figure 4 : The force-displacement curves for samples A and B, plotted on the same axes. Sample A supports a higher load and has a higher spring constant than sample B

Engineering Stress and Engineering Strain

Video

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The solution to our problem with force and displacement being dependent on sample size is to normalize by the dimension over which the force or the length change is being distributed. The force is being supported by the cross- sectional area normal to the applied load, so we define the engineering stress as

$$\sigma = \frac{F}{Ao} \tag{3}$$

where F is the applied load and A0 is the initial (unloaded) cross-sectional area, as illustrated in Figure 5.

EXERCISE YOUR KNOWLEDGE

Q2.1.1
Review

Which of the following is the best explanation of the engineering stress?

Select an answer and submit. For keyboard navigation, use the up/down arrow keys to select an answer.

a	The applied force divided by the initial cross-sectional area
b	The applied force divided by the instantaneous cross-sectional area
c	The applied force divided by the cross-sectional area within the necked region
d	None of the above

Correct Answer:

✓ a - The applied force divided by the initial cross-sectional area

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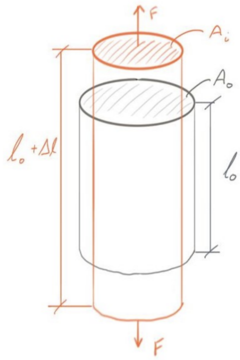

Figure 5: A cylinder "streched" along its long axis. Note that the cross sectional area decreases while under load (this is the orange cylinder)

You may know that the cross-sectional area will decrease when "stretched" but we don't need to account for this yet.

Just as a thicker sample supports more load or needs more load to be made to elongate a certain amount, so does the absolute change in length depend on how long the sample is. So, instead of using displacement, we need to use strain and we define engineering strain as

$$\varepsilon = \frac{\Delta l}{l_0} \qquad (4)$$

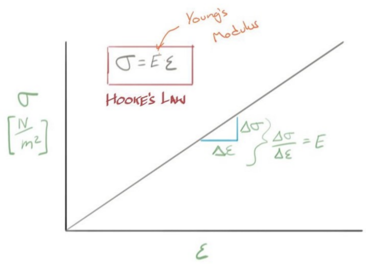

Figure 6: A general linear stress versus strain curve.

Now, with great excitement, we can plot the stress and strain for samples A and B, confident that the curves will indicate the behaviour of the material itself, without influence from the sample size. The resulting curve is shown in Figure 6. The slope of this line is given the capital letter "E" and is called the Young's modulus. Now we can write Hooke's Law again in terms of stress and strain

$$\sigma = E\varepsilon \qquad (5)$$

So nice.

EXERCISE YOUR KNOWLEDGE

Q2.2.1
Review

Aluminum has a Young' s modulus of 69 GPa. If a stress of 60 kPa is supported by an aluminum component what will the resulting strain in the component be? Express your final answer in scientific notation with 2 significant figures and do not include units in your final response. Note: it is important that you format your response exactly as indicated, otherwise your response may be graded as incorrect. To write 1.1×10^{-7} write 1.1e-7. Note that there are no spaces.

Type your answer and submit

Correct Answer:

✓ 8.7e-7

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Q2.3.1(a)
Review

Two cylindrical titanium bars are loaded along their long axes in tension, each by a load of 1.2 MN. Bar A has a radius of 5 cm and length of 4.5 m while bar B has a radius of 2.5 cm and a length of 9 m. The Young' s modulus of titanium is 120 GPa. (a) Calculate the stress in bar A in MPa. Express your final answer with 2 significant figures and do not include units in your final answer.

Type your numeric answer and submit

Correct Answer:

✓ 153.0

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Q2.3.1(b)
Review

(b) Calculate the stress in bar B in MPa. Express your final answer with 2 significant figures and do not include units in your final answer.

Type your numeric answer and submit

Correct Answer:

✓ 611.0

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Q2.3.1(c)
Review

(c) Calculate the elongation, in mm, that will occur in bar A while this load is applied. Express your final answer to 2 significant figures and do not include units in your answer.

Type your numeric answer and submit

Correct Answer:

✓ 5.7

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Q2.3.1(d)
Review

(d) Calculate the elongation, in mm, that will occur in bar B while this load is applied. Express your final answer to 2 significant figures and do not include units in your answer.

Type your numeric answer and submit

Correct Answer:

✓ 46.0

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Q2.3.1(e)
Review

(e) Calculate the force, in kN, that would need to be applied to bar B, such that it would experience the same stress as bar A while loaded with a 1.2 MN force. Express your final answer to 1 significant figure and do not include units in your answer.

Type your numeric answer and submit

Correct Answer:

✓ 300.0

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Some Possible Ways to Define Elastic Deformation

Let's take a stab at defining elastic deformation. Probably the first thing that would come to mind would be that the thing you are loading doesn't change shape permanently, or more formally we could say,

> *Potential definition 1: During elastic deformation the sample dimensions return to their original dimensions upon unloading.*

This definition isn't too bad. It's clear and fairly easy for us to visualize in our mind's eye. However, it practice it is actually difficult to use since it depends on our ability to measure the sample dimensions accurately and this can introduce a lot of error. How about this definition that is really an extension of the first, but moves the emphasis away from the macroscopic dimensions:

> *Potential definition 2: During elastic deformation atoms return to their original positions upon unloading.*

This definition isn't too bad either. It draws attention to the importance of atoms and their arrangement. It is also fairly easy for us to build a model in our mind's eye of some atoms being pulled apart and then popping back to their original positions. However, it is again impractical since it relies on our ability to visualize individual atoms, or features close to that level. Both of these definitions have merit and are generally correct, they are just difficult to put into practice.

EXERCISE YOUR KNOWLEDGE

Q2.6.1
Review

Which of the following are possible definitions of elastic deformation? Choose all correct options.

Multiple answers: Multiple answers are accepted for this question

Select one or more answers and submit. For keyboard navigation... SHOW MORE

a	Atoms do not move to new equilibrium positions
b	Atoms move to new equilibrium positions
c	The sample does not return to its original dimensions upon unloading
d	The sample returns to its original dimensions upon unloading

Correct Answers:

- ✓ **a** - Atoms do not move to new equilibrium positions
- ✓ **d** - The sample returns to its original dimensions upon unloading

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A Simple Model for Bonding in a Solid

I'm going to go out on a limb here and make the assumption that, since you are reading this, you believe in the existence of atoms. Incidentally, many parts of thermodynamics can be learned and used without the need to believe in atoms. More on that later.

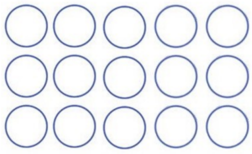

Figure 7: Atoms represented as circles, at equilibrium.

Let's use a simple model here. Let's imagine that atoms are hard spheres that we can stack up next to one another. Think of a box full of ball bearings. Let's call it the hard sphere model. Okay, somebody else already did, I can't make this stuff up. We'll use the hard sphere model more in the section on crystal structures of solids. Think of fifteen of these hard spheres in a solid at rest, as in Figure 7. If we apply a load to this solid and only apply enough load to ensure that the deformation stays within the elastic region we can infer that the atoms must move back to their original positions once we remove the load. We know this because we know that the sample dimensions have not changed, because that was one of our definitions of elastic behaviour.

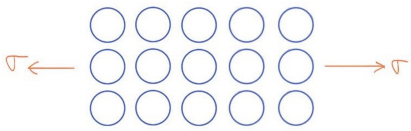

Figure 8: Atoms represented as circles, under elastic load.

We can also reason out that while the load is applied, since the sample gets longer, the atoms that make up that solid must also move away from one another, as in Figure 8. If we release the load the atoms return to their original positions, as in Figure 7. (As an aside, we are not yet concerned with exactly what the arrangement of atoms is in a solid. By sketching the atoms in such an organized manner I have hinted at the regular repeating arrangement of atoms in ordered solids, but the accuracy of my sketch in 2 dimensions is not important at this point.)

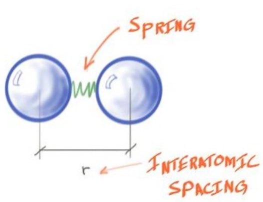

Figure 9: Atoms as spheres connected by a spring. Note that the dimension r is the distance from the centre of one atom to the centre of the next atom. (Do you like the shading on the spheres?)

What if we modeled atoms in a solid as spheres connected to one another by little springs, as shown in figure 9? This is actually not a bad model for the net force between atoms in a solid. There is an attractive force that pulls the atoms together, but then as the atoms get closer to each other there is a repulsive force that becomes significant. If we plotted the net force, that is, the sum of the attractive and the repulsive forces we would get a curve that would look similar to figure 10.

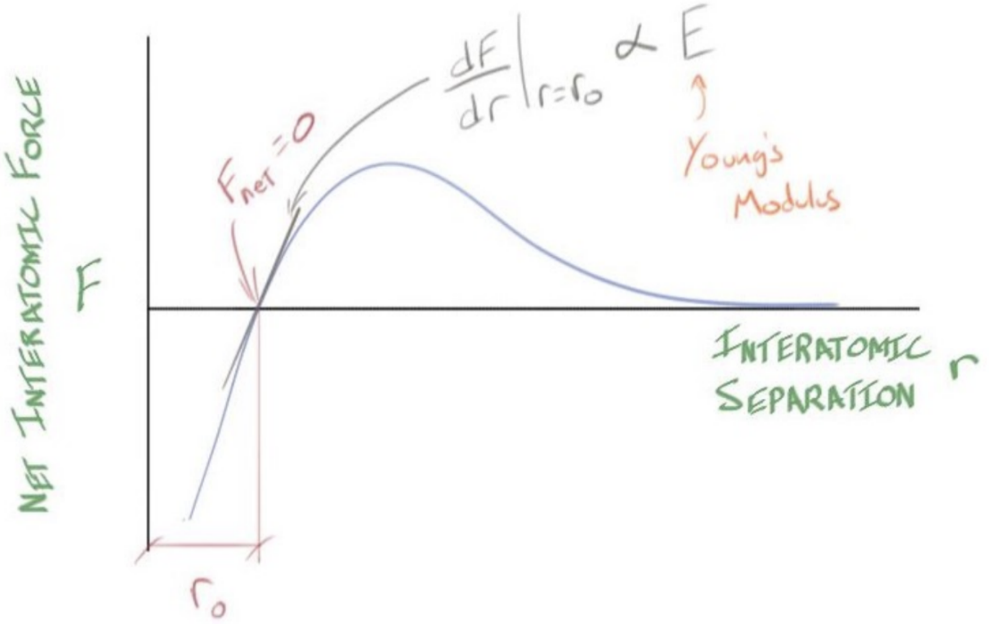

Figure 10: Interatomic force versus interatomic separation curve for two hypothetical atoms

If you look closely at the curve in figure 10 you'll notice an important point: the point where the net force is equal to zero. This corresponds to the atoms at rest and so the value of the interatomic spacing at that point defines the so called, equilibrium interatomic spacing. It is worth noting that the letter "r" is used for interatomic spacing, but should not be confused with the radius. Yes, this is confusing, but it is a convention so we need to smile and roll with it.

The Structure Independence of Young's Modulus

Video

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We can reason something else out from carefully considering Figure 10. Let's consider small disturbances of atoms from their equilibrium interatomic spacing, such as the loads that would lead only to elastic deformation. We apply a force to atoms and observe a change in the spacing between the atoms. Considering the spring model of Figure 9 again it may become more obvious that what we are really considering is a plot of force versus displacement. That sounds familiar! Oh, yes, that was Hooke's law and the spring constant.

So, it shouldn't be a huge leap then to imagine the force required to pull two atoms slightly apart is somehow related to the force required to elongate a macroscopic sample. What might this relationship be: linear, quadratic, logarithmic, or something else? In fact, the Young's modulus is directly pro- portional to the spring constant of the tiny little interatomic spring, which we could right as follows

$$E \propto \left.\frac{dF}{dr}\right|_{r=r_0} \tag{6}$$

EXERCISE YOUR KNOWLEDGE

Q2.8.1
Review

Permanently deforming a metal will change its Young' s modulus.

Select an answer and submit. For keyboard navigation, use the up/down arrow keys to select an answer.

a	True
b	False

Correct Answer:

✓ b - False

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Example: Materials Substitution: synthetic woodwind reeds

Professor Mark Kortschot

When we use a material to make a product, we only really care about the collection of material properties it has, not its structure. Of course, the structure determines the properties, *but it is possible to find two or more materials that have very different structures and yet similar properties. When this is true, we often find competing products made from "competing" materials. For example, ladder frames and steps are made from both aluminum and fiberglass (a mixture of glass fibers and polymer resin). Banknotes are historically made from paper, but increasingly printed on polyester film.*

A good example of materials substitution may be found in the world of music. Woodwind reeds are essentially thin vibrating beams that are attached to a mouthpiece to generate the tones in clarinets and saxophones. Traditionally, reeds are made from Arundo Donax, a fast-growing species of cane, which is similar to bamboo. This material has a combination of low density and high stiffness that makes it perfect for the job of vibrating at hundreds or thousands of cycles per second. But the performance of cane reeds is highly variable, since they come from a natural resource, and they are susceptible to changes in humidity and temperature. They also break down relatively quickly and need to be replaced. As a result, many people have tried to find alternatives to cane, but a replacement material, if it is to be used with reeds of conventional shape, must match both the low density and high longitudinal modulus of traditional cane.

The trouble with most polymers and metals is that they are too dense for this application, and a heavy reed simply doesn't vibrate in the same way that a light cane reed vibrates. Some commercial synthetic reeds use a combination of polymers and glass or carbon fibres (for stiffness). These composites can be stiff enough but are also denser than cane, unless hollow elements are included. In 1998, Legere Reeds Ltd., a Canadian company, introduced synthetic reeds made of oriented polypropylene. This material has a density of 0.91 g/mL, which matches that of wet cane quite closely. Ordinary polypropylene is too flexible (low elastic modulus), but Legere stretches the polypropylene to line up the covalent backbones of the molecules, resulting in a very high stiffness, which can be precisely adjusted. (Note all woody materials grow naturally with partially aligned cellulose cellulose molecules, and this is what makes cane so stiff in the first place.) Legere can cut conventional reed shapes from this new material, and the resulting reeds perform well, but are less variable and far more durable than cane reeds, making them popular on Broadway and in many of the world's top orchestras.

Many new businesses have been founded based on a simple substitution of a novel material for a conventional one in an existing application. The key is not to try to match the structure of the incumbent material, but to match all of the material properties that are important for the application.

How Do We Actually Get a Stress-Strain Curve?

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So, now we know quite a bit about elastic behaviour. We've seen that most metals have an initial linear elastic region on their stress-strain curves. But how do we actually get the stress-strain curve in the first place? You may think, well, could we use a cylinder like in Figure 5? The problem with that is that it is difficult to grip a sample like that hard enough to stretch it (load in tension) without significantly deforming the ends where we are gripping it. To solve this problem we use a tensile specimen, also called a tensile coupon, or if you want to be hip, a dog-bone specimen. These tensile specimens may be either cylindrical or rectangular in cross-section.

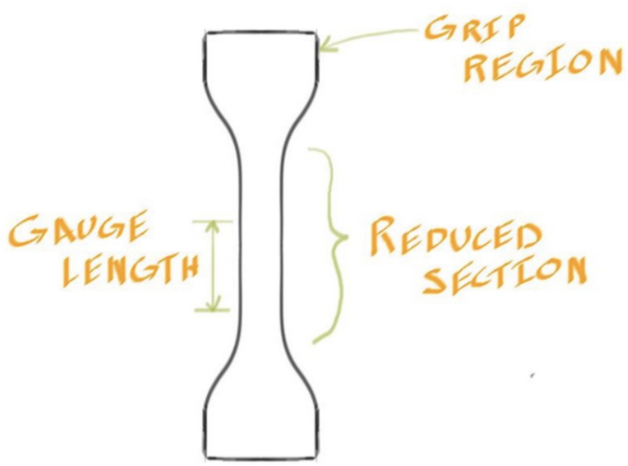

Figure 11: A cartoon sketch of a typical tensile specimen showing the grip region, reduced section and gauge length.

Figure 11 shows a typical tensile specimen with a few important features labelled. The grip regions are probably fairly self explanatory; this is where we grip the sample. The force supported by the sample is consistent along its length, but because of the larger cross-sectional area, the stress in the grip regions is lower, so we can squeeze it really hard without worrying about impacting the results of the test. The region through the middle with parallel sides, but a smaller cross- sectional area is called the reduced section, since the cross-sectional area is, wait for it, reduced! Now, a final point to discuss is how we establish the initial length of the sample l_0 to use in our engineering strain equation. For tests that you aren't planning to publish or rely on for important designs it may be okay to just use the entire length of the reduced section, however, this includes the rounded regions where the cross-sectional area is varying from the grip region to the reduced section. It is also difficult to accurately measure the elongation, so for a proper tensile test, we typically use a so-called strain gauge. This is a rather expensive piece of electronic equipment that clips on to the sample in the reduced section and very accurately measures the strain. The initial length of the strain gauge is where we define the initial length l0. There are also optical systems that can do this without the need to touch the sample. One advantage of these is that the sample can be loaded all the way to fracture. If you let a sample fracture with a conventional strain gauge on t the fracture event will often damage the strain gauge. That's bad. Did I mention that they can be quite expensive?

EXERCISE YOUR KNOWLEDGE

Q2.4.1
Review

Which of the following is not a part of a conventional uniaxial tensile test?

Select an answer and submit. For keyboard navigation, use the up/down arrow keys to select an answer.

a	Powerful grips to hold the sample
b	A load cell
c	Supports on which to rest the sample
d	An accurate means to determine strain

Correct Answer:

✓ c - Supports on which to rest the sample

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Q2.5.1
Review

Regarding a standard tensile specimen, which of the following statements is true?

Select an answer and submit. For keyboard navigation, use the up/down arrow keys to select an answer.

a	The gauge length is normally longer than the reduced section, but the reduced section is never longer than the gauge length
b	The gauge length is normally less than the reduced section, but the reduced section is never less than the gauge length
c	The reduced section and the gauge length are normally equal
d	None of the above

Correct Answer:

✓ b - The gauge length is normally less than the reduced section, but the reduced section is never less than the gauge length

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Wrapping it all up

What have we learned in this section? Well, elastic deformation involves atoms moving slightly away from their equilibrium positions. The Young's modulus describes elastic behaviour. Finally, we test the mechanical behaviour of a material by creating a tensile specimen that experiences stress in the reduced section as it is deformed in tension.

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